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Periodic orbits and semiclassical quantization of dispersing billiards

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Abstract. Periodic orbits in a dispersing billiard system consisting of three circular arcs are studied numerically by using a partial coding rule together with an efficient method for enumerating periodic orbits on the real billiard plane. By examining several statistical measures, it is shown that the length spectrum and the stability exponents are highly uncorrelated. The validity of the semiclassical trace formula is also tested, and a remarkable agreement of the semiclassical and quantum density of states is obtained at least for about the lower 15 levels.

1. Introduction

Dynamical systems with more than one degree of freedom demonstrate surprisingly complicated behaviour, an understanding of which is well developed at least in classical mechanics. Recent advance in the study of classical dynamical systems reveals that there are distinctive or hierarchical classes such as ergodicity, mixing and the Bernoulli property, even within systems showing chaotic behaviour [1]. On the other hand, a full understanding of the quantum system whose classical counterpart displays chaos is far from accomplished. Even when we restrict ourselves to strongly chaotic or ergodic systems where no KAM tori or islands exist in the phase space, we have not yet found a complete answer to various conjectures regarding the universality of level statistics, characteristic nature of wave functions, and so on [2]. Hence, before analysing generic Hamiltonian systems with complicated phase space structure, we should investigate the extremely chaotic case as a first step.

The most naive approach to this problem would be to find a clear translation rule from the language of classical mechanics to that of quantum mechanics. For this purpose, the semiclassical method usually plays an important role and provides us with a powerful tool. In particular, Gutzwiller's trace formula which relates a set of classical periodic orbits including their stability to the quantum energy levels, has been a central subject in the analysis of this issue [3]. However, due to our lack of knowledge, even in the numerical sense, concerning periodic orbits of chaotic systems, it is difficult to get a clear perspective on the quantum–classical correspondence in strongly chaotic systems. To attack this kind of difficulty, there are be two different approaches. One is to obtain analytically a complete set of periodic orbits by choosing an appropriate model [4–7], and the other is to calculate periodic orbits numerically by brute force [8, 9]. The drawback of the former approach is that due to the specificity of the model system, the resulting length spectrum often becomes non-typical, although it gives a precise asymptotic behaviour that we really know. Since the numerical calculation can handle only finite data, the latter approach cannot generate all periodic orbits, but it keeps the generality of the system.

In the present article, we will take the latter approach by studying a dispersing billiard problem [10]. The advantage of the billiard problem is that the classical properties are easily controlled by designing the shape of the boundary, and that periodic orbits can be obtained by geometrical speculation. In particular, our dispersing billiard system introduced in section 2 has several system parameters which do not change the category of classical dynamics. It enables us to explore universal aspects of periodic orbits independent of a special shape of the boundary. In addition, as is shown in section 3, our dispersing billiards have a rather good correspondence between a sequence of codes and a periodic orbit, which markedly excludes the possibility of missing periodic orbits. Combining this nice property with the efficient numerical method proposed below, we have succeeded in obtaining several thousand periodic orbits, and examined their statistical properties from various viewpoints. Using these periodic orbits, we also tested the validity of the semiclassical trace formula for this type of dispersing billiard and discuss the problems posed by the present analysis.

2. The dispersing billiards

The billiard problem in classical mechanics is the study of the motion of a particle which moves freely within a region D with a constant energy and satisfies the law of reflection at the boundary ∂D, the angle of incidence being equal to the angle of reflection. The corresponding quantum mechanical problem is the study of eigenvalues and eigenfunctions of the time-independent Schrödinger equation which describes the stationary states of a particle in D

$$\frac{\hbar^2}{2m}\nabla^2\psi(q) = E\psi(q) \qquad q = (x, y) \in D \tag{2.1}$$

with Dirichlet boundary condition,

$$\psi(q) = 0 \qquad q \in \partial D. \tag{2.2}$$

The dispersing billiard which we shall be interested in here is a class of billiard systems having boundaries with curvature measured by inner normal positive everywhere. The boundary of our model billiards consists of three circular arcs Γ_0, Γ_1 and Γ_2 as shown in figure 1. Each arc intersects at vertices A, B and C of a triangle, and the centre of each circle, which is not shown in figure 1, is given as the vertex of an equilateral triangle. The independent geometrical parameters which we can control are the length R of the base, the base angles ϕ_{20}, ϕ_{01} of a triangle ABC, and the base angles α_0, α_1 and α_2 of the equilateral triangles. The reasons for employing this type of dispersing billiard are mainly twofold; (1) Several rigorous results concerning the corresponding classical dynamics have already been derived. (2) Generic or universal properties can be extracted by studying several sets of system parameters systematically. Our main aim in this paper, therefore, is to analyse properties

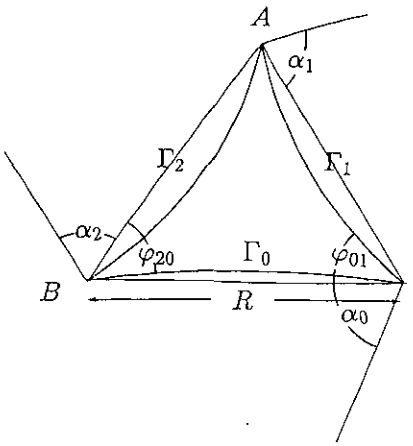


Figure 1. Schematic picture of a dispersing billiard system whose boundary consists of three circular arcs.

of periodic orbits and the validity of the semiclassical theory of such classically well examined systems in a systematic manner, which has never been done before.

This shape of billiard includes several characteristic cases as particular sets of system parameters. For example, if the circular arcs meet tangentially, the power law decay of the auto-correlation function for the velocity of classical trajectories is observed [11], and if the radius of the three circles are infinite, it becomes a polygonal billiard system which does not show exponential instability despite its non-integrability [12, 13]. However, our present concern is not to study such special classes, but to examine a *generic hyperbolic case not having an anomalous nature in classical dynamics*. As is rigorously shown, this class of dispersing billiards have ergodic property and is a K-system [14]. Furthermore, it is isomorphic to the Bernoulli shift [15]. This means that the Lyapunov exponent of the system is positive. An important point is that the method for constructing the Markov partition has been obtained, and it shows that a countable but infinite number of partitions are necessary [16, 17]. It makes it difficult to code periodic orbits using a finite number of symbols, which prevents us from directly applying the curvature expansion method proposed by Cvitanovic and Eckhardt [18]. As is naturally expected from the properties mentioned above, the number of periodic orbits proliferates exponentially. There is actually a rigorous proof which shows that both the lower and upper bound of such a proliferation rate is given as an exponential function [17].

3. Coding of periodic orbits

The most favourable point of our billiard system is that there exists a rather simple way of coding each periodic orbit, which considerably diminishes our labour finding periodic orbits, and ensures the upper bound of the number of periodic orbits when the bounce number is given. Following the argument made by Morita [19], we here show that each periodic orbit γ can be coded uniquely by the symbolic sequence $\xi(\gamma) \in \Sigma$, where $\Sigma = \{\xi = (\xi_j)_{j=-\infty}^{j=\infty} \in \prod_{j=-\infty}^{\infty}\{0,1,2\} \mid \xi_j \neq \xi_{j+1} \text{ for any } j\}$, and we represent the bounce at each circular arc Γ_0, Γ_1, and Γ_2 by the symbol 0, 1 or 2.

First consider an orbit which reflects at a point P on the arc Γ_1, and then reflects at a point P_1 on the arc Γ_2. Define r as the distance between the points P and a vertex C measured along the arc Γ_1 and ϕ as the angle between the inner unit normal vector and the momentum vector going out from the point P_1. r_1 and ϕ_1 are defined in the same manner. From geometric consideration, one can obtain

$$\frac{\mathrm{d}r_1}{\mathrm{d}r} = -\frac{\cos\phi}{\cos\phi_1}\left[1 + \frac{\tau}{\cos\phi}\left(\frac{\mathrm{d}\phi}{\mathrm{d}r} + k(r)\right)\right] \tag{3.1}$$

where τ is the Euclidean distance between the points P and P_1 and $k(r)$ is the geometric curvature of the arc Γ at the point P. For the consecutive reflections, $(r_0, \phi_0), (r_1, \phi_1), \ldots, (r_n, \phi_n)$, we obtain

$$\frac{\mathrm{d}r_n}{\mathrm{d}r_0} = \frac{\mathrm{d}r_n}{\mathrm{d}r_{n-1}}\frac{\mathrm{d}r_{n-1}}{\mathrm{d}r_{n-2}}\cdots\frac{\mathrm{d}r_1}{\mathrm{d}r_0} = (-1)^n\frac{\cos\phi_0}{\cos\phi_n}\prod_{j=0}^{n-1} b_j \tag{3.2}$$

where

$$b_j = \left[1 + \frac{\tau_j}{\cos\phi_j}\left(\frac{\mathrm{d}\phi_j}{\mathrm{d}r_j} + k(r_j)\right)\right]. \tag{3.3}$$

Similar geometrical consideration yields

$$\frac{\mathrm{d}\phi_1}{\mathrm{d}r_1} = k(r_1) + \cos\phi_1\left[\tau + \cos\phi\left(\frac{\mathrm{d}\phi}{\mathrm{d}r} + k(r)\right)^{-1}\right]^{-1}. \tag{3.4}$$

$\mathrm{d}\phi_j/\mathrm{d}r_j$ $(j = 1, 2, \ldots, n)$ is non-negative if $\mathrm{d}\phi_0/\mathrm{d}r_0 \geqslant 0$ is satisfied. Since $\cos\phi_i \geqslant 0$ and $k(r_j) \geqslant 0$, there exists $\eta > 0$ such that

$$b_j \geqslant 1 + \eta \qquad j = 0, 1, \ldots, n-1. \tag{3.5}$$

Hence, in order to make two different orbits which are initially on the same arc trace the same history of codes within n steps, the following relation must be satisfied

$$r(P, P') \leqslant |\cos\phi_0|^{-1}\, l(1+\eta)^{-n} \tag{3.6}$$

where $r(P, P')$ is the initial distance between different points P and P' and l is the minimum arc length. Because we are now concerned only with periodic orbits, this inequality leads us directly to the conclusion that the order of reflections at the arcs, or the sequence of codes, determines a unique periodic orbit. Of course, in this argument, it is implicitly assumed that there exist no periodic orbits passing tangentially a point on the arc. Even when such periodic orbits exist, a slight deformation of the boundary changes them into non-tangent ones. It ensures the uniqueness shown here. At this point, we should remark that this correspondence between a sequence of codes and a periodic orbit on the billiard plane is not necessarily one-to-one, while there exists a strict one-to-one correspondence between them in the case of non-compact dispersing billiards [20]. In calculations performed below, we actually found many sequences of codes which do not have a corresponding periodic orbit.

4. The method of finding periodic orbits

The procedure of searching for periodic orbits is divided into two parts. The first task is to generate all the symbolic sequences without omission, and the second one is to find the corresponding periodic orbits on the real billiard plane. One effective method for generating the symbolic sequence $\xi(\gamma)$ is as follows: first we prepare the binary expression for 2^n integers, $K = 0, 1, \ldots, 2^n - 1$, i.e. $K = a_1 \times 2^{n-1} + a_2 \times 2^{n-2} + \cdots + a_n \times 2^0$, and then generate the sequence $\sigma_1\sigma_2\ldots\sigma_n$ by the rule given as $\sigma_i = \sum_{j=1}^{i}(a_i + 1) \pmod 3$. By construction, this sequence satisfies the prohibition rule as $\sigma_i \neq \sigma_{i+1}$. The second step is to discard the sequences not satisfying the condition $\sigma_0 = \sigma_N$, and finally the sequences obtained by the permutation of others or the repetition of smaller periodic sequences are excluded.

Once a symbolic sequence $\sigma_1\sigma_2\ldots\sigma_n$ is given, the remaining procedure to obtain periodic orbits on the real billiard plane is to find a closed polygon whose vertices are on the corresponding arcs, and at the same time its perimeter is the smallest of all possible configurations. The reason for the second requirement is that since the curvature of the boundary is everywhere positive, there are no conjugate points on the billiard table. It is, of course, equivalent to the law of reflection. More precisely, a desirable polygon must satisfy the following set of equations

$$\frac{\partial}{\partial r_{\sigma_j}}(L_{\sigma_{j-1}\sigma_j} + L_{\sigma_j\sigma_{j+1}}) = 0 \tag{4.1}$$

where $L_{\sigma_j\sigma_{j+1}}$ is the distance between the points P_{σ_j} and $P_{\sigma_{j+1}}$ and r_{σ_j} is the distance between the point P_{σ_j} and the point C measured along the boundary. These equations are usually solved by a Newtonian method [21]. However, in the present model, there exists a number of spurious solutions which satisfy the condition of extremum, but do not yield true periodic orbits. They correspond to stationary polygons, at least one of whose vertices is outside the billiard boundary while each vertex of such stationary polygons is located on one of the three circles constituting the boundary. These stationary polygons satisfy the minimax condition instead of the minimum one. Therefore, unless initial points are incidentally located near a true solution, much computing time is required to get a final solution. To avoid this, we use the following technique.

We first put a set of points $\{P_{\sigma_j}^{(0)}\}$ on the corresponding arc Γ_{σ_j}. The points $P_{\sigma_0}^{(0)}$ and $P_{\sigma_2}^{(0)}$ being fixed, $P_{\sigma_1}^{(1)}$ is determined by solving equation (4.1) for the case $j = 1$ using a Newtonian method. As is shown in figure 2, because $P_{\sigma_1}^{(1)}$ always exists on the arc XY, solving equation (4.1) is not difficult. In the same manner, $P_{\sigma_1}^{(0)}$ and $P_{\sigma_3}^{(0)}$ being fixed, $P_{\sigma_2}^{(1)}$ is determined. This procedure is repeated until $P_{\sigma_0}^{(1)}$ is determined. By considering this procedure as one step, we make the next-step optimization starting from the preceding configuration $\{P_{\sigma_j}^{(1)}\}$, and then obtain a set of more optimized points $\{P_{\sigma_j}^{(2)}\}$. As points $\{P_{\sigma_j}^{(n)}\}$ are always located on the boundary of billiards, or three circular arcs, one does not suffer from spurious solutions mentioned above. From the preceding argument, the converged points $P_{\sigma_j}^{(\infty)} = \lim_{n\to\infty} P_{\sigma_j}^{(n)}$ form periodic points which correspond to a given sequence of codes $\{\sigma_j\}$. In our experiments, when the maximal displacement between successive steps, i.e. $\max_{\sigma_j} \mid P_{\sigma_j}^{(n)} - P_{\sigma_j}^{(n-1)} \mid$ is smaller than $\epsilon = 10^{-9}$, we judge the points

$\{P_{\sigma_j}^{(n)}\}$ to have converged. In actual computations, the upper limit of the step number n is set at 1000. In several cases where there exists no periodic orbit corresponding to the symbolic sequence, the points P_{σ_j} and $P_{\sigma_{j+1}}$ converge slowly to the same point at which the arcs Γ_{σ_j} and $\Gamma_{\sigma_{j+1}}$ intersect one another. Therefore, we regard such a symbolic sequence as having no corresponding periodic orbit, and we give the criterion for halting our search as $L_{\sigma_j\sigma_{j+1}} < 10^{-6}$.

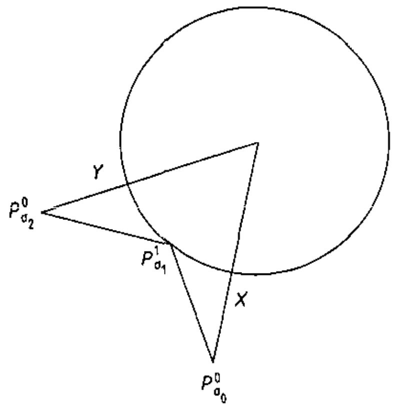
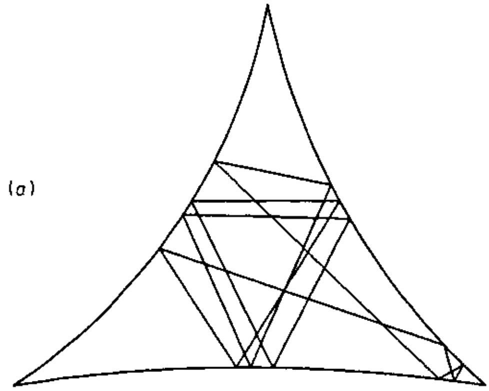
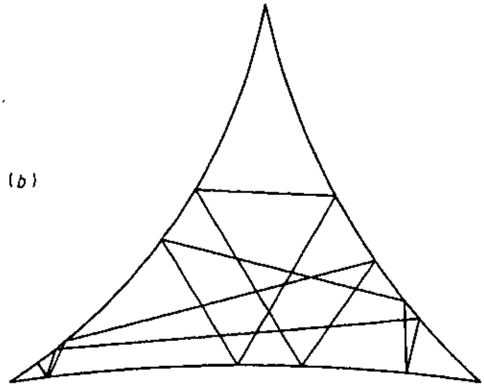


Figure 2. A stationary point $P_{\sigma_1}^{(1)}$ on the arc XY.

5. The property of periodic orbits and its universality

Using the method explained in the preceding section, we examined several cases by changing the value of an angle α_2 from $\frac{10}{21}\pi$ to $\frac{10}{27}\pi$ with other system parameters $R = 0.01$, $\phi_{01} = \frac{1}{3}\pi$, $\phi_{20} = \frac{13}{42}\pi$, $\alpha_0 = \frac{5}{11}\pi$ and $\alpha_1 = \frac{2}{5}\pi$ being fixed. The minimum reflection number of periodic orbits in this billiard system is 3, and the maximal period we searched is 20. Within the above criterion, the percentage of symbolic sequences whose convergency cannot be judged is approximately 1%, and several thousand primitive periodic orbits were eventually obtained. The result is tabulated in table 1. There are, in general, two types of periodic orbits; the first type is a class of orbits which retrace their own trajectories in the reverse direction before they return back to their initial configuration. The second type are ones not tracing their own trajectories before they close. Therefore, the sequence of codes for periodic orbits of the first type is necessarily symmetric about a central code. An example of this is illustrated in figure 3(*c*). On the other hand, the sequence of codes for the second type do not have such a symmetry. Typical examples are actually presented in figures 3(*a*) and 3(*b*). Using the present method of generating the sequence of codes and for the periodic orbits of the second type, two different codes, either one of which is reversed to give the other, yield periodic orbits with the same topology on the billiard plane. In order to avoid the double-counting of periodic orbits with the same topology, these pairs of periodic orbits of the second type are identified and the number of periodic orbits with different topology are listed in table 2. Because the curvature of Γ_2 increases as α_2 decreases, neighbouring orbits diverge more rapidly due to the reflection at this boundary Γ_2, and thus the system exhibits stronger chaos. This means that the variety of periodic orbits with different topology,





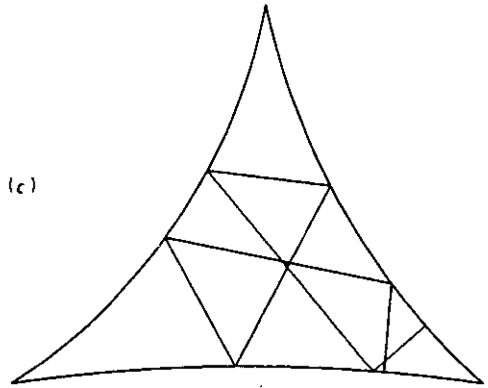


Figure 3. Examples of periodic orbits, each of which respectively corresponds to the symbolic sequence:
(*a*) $\{1,2,0,1,2,0,1,2,0,1,2,0,1,0\}$.
(*b*) $\{1,2,0,1,2,0,1,2,0,2,0,2,1,0\}$.
(*c*) $\{1,2,0,1,2,0,1,0,2,1,0,2,1,0\}$.

which correspond to the symbolic sequences not having corresponding periodic orbits at larger values of α_2, is increased. This is the reason why in table 1 the number of periodic orbits increases as α_2 decreases.

For the latter set of primitive orbits shown in table 2, we first examined the cumulative density of length spectrum, $N(l) = \sharp\{$ primitive periodic orbits γ with length $l_\gamma \leqslant l\}$. The result is shown in figure 4. In the present procedure, periodic orbits are obtained in order of the reflection number, while the order of reflection number is not necessarily the same as that of their length. Accordingly, to observe the precise behaviour of the cumulative density, we employed the periodic orbits ranging from 50 to 8000 in order of their length. In this range, there were no omissions of periodic orbits as the order of length. As is expected from the result of the systems with a finite number of Markov partitions [19,20], $N(l)$ can be approximated very well by the exponential curve asymptotically. Figure 4 provides clear evidence for this prediction. Values of h which is the exponent of the exponential distribution, $N(l) = \text{constant} \times e^{hl}$, are shown in figure 5. In this expression, h represents topological entropy which measures the variety of orbits. From the relation between the value of α_2 and the number of periodic orbits, we can understand that in figure 5 h increases as α_2 decreases. This exponential proliferation law is of course the crudest information about the length spectrum, and it directly reflects the chaoticity of classical dynamics. However, except for the work of Sieber and Steiner [9], little is known regarding fluctuation properties of the length spectrum for the planer billiard.

Table 1. Numbers of periodic orbits.

n	$\pi/2.1$	$\pi/2.2$	$\pi/2.3$	$\pi/2.4$	$\pi/2.5$	$\pi/2.6$	$\pi/2.7$
3	2	2	2	2	2	2	2
4	3	3	3	3	3	3	3
5	6	6	6	6	6	6	6
6	1	4	5	7	8	8	8
7	6	10	10	10	14	16	16
8	11	13	17	17	18	20	22
9	20	22	26	36	36	38	38
10	25	36	46	60	61	62	68
11	50	66	82	92	100	110	122
12	82	102	135	151	167	192	205
13	136	188	248	280	318	350	378
14	228	311	406	492	569	613	676
15	368	528	706	862	1014	1118	1208
16	601	895	1223	1498	1780	1992	2186
17	1022	1568	2158	2698	3176	3606	3998
18	1710	2701	3751	4819	5685	6477	7236
19	2888	4638	6592	8598	10294	11782	13040
20	4850	7984	11605	15226	18525	21365	23735
	12009	19077	27021	34857	41776	47760	52947

Table 2. Numbers of periodic orbits (orbits without time reversal symmetry identified).

n	$\pi/2.1$	$\pi/2.2$	$\pi/2.3$	$\pi/2.4$	$\pi/2.5$	$\pi/2.6$	$\pi/2.7$
3	1	1	1	1	1	1	1
4	3	3	3	3	3	3	3
5	3	3	3	3	3	3	3
6	1	4	5	7	8	8	8
7	3	5	5	5	7	8	8
8	7	9	11	11	12	14	16
9	10	11	13	18	18	19	19
10	19	26	33	41	42	43	46
11	25	33	41	46	50	55	61
12	49	62	79	89	99	115	124
13	68	94	124	140	159	175	189
14	133	181	233	279	323	351	387
15	184	264	353	431	507	559	604
16	330	488	664	811	960	1074	1180
17	511	784	1079	1349	1588	1803	1999
18	914	1432	1975	2529	2982	3400	3794
19	1444	2319	3296	4299	5147	5891	6520
20	2521	4136	5984	7829	9519	10980	12193
	6226	9855	13902	17891	21428	24502	27155

If we wish to understand, by using Gutzwiller's trace formula, the universality of the quantum spectrum from that of the classical length spectrum, we must know more detailed information with respect to the generic features of the periodic orbits. As mentioned in the introduction, the advantage of our dispersing billiard is that a generic property of the length spectrum can be inferred by changing several system parameters in various ways.

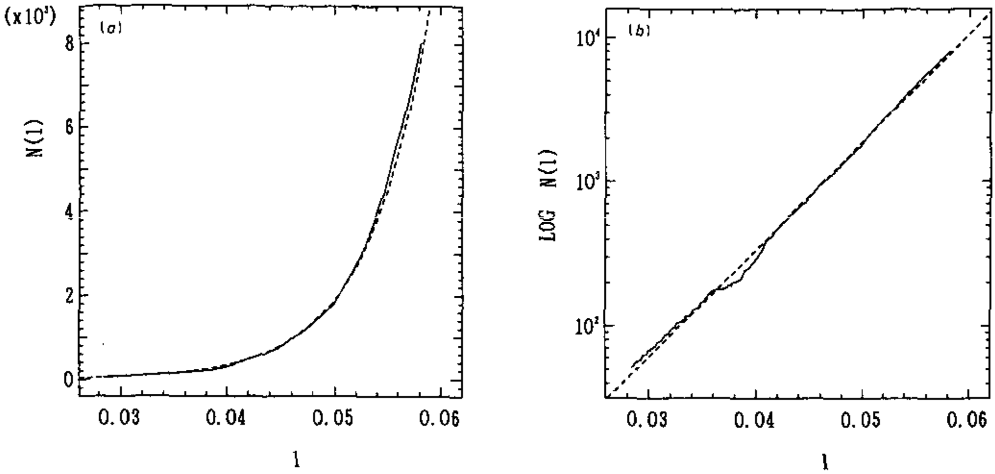


Figure 4. Cumulative density function $N(l)$ of the length spectrum in the case of $\alpha_2 = \pi/2.4$ fitted by the exponential proliferation law, e^{hl} for $h = 1.723 \times 10^2$. (*a*) linear scale. (*b*) logarithmic scale.

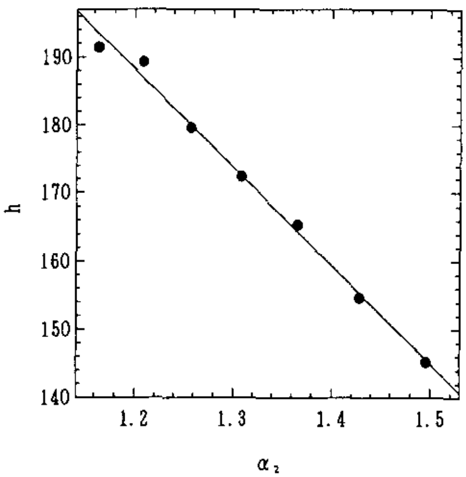


Figure 5. Topological entropy $h(\alpha_2)$.

As a first step in exploring such a generic feature, we studied the nearest neighbour spacing distribution of the length spectrum. By following the procedure developed in the study of quantum energy spectra, we use the averaged cumulative density obtained numerically to unfold the original spectrum. The result is shown in figure 6. The numerical histograms agree surprisingly well with the exponential curve, or Poisson distribution, which implies that the length of periodic orbits are uncorrelated at least within the short range. Every set of system parameters we have studied yield a Poisson distribution as shown in figure 6 though the length spectrum itself depends on the system parameter and is quite different from one to another. Therefore, we conjecture that this property is universal for all hyperbolic billiard systems.

In order to check, from the statistical viewpoint, how far the length spectrum of strongly chaotic systems is correlated or uncorrelated, we must know information about higher-order correlations for the length spectrum. In the analysis of the energy

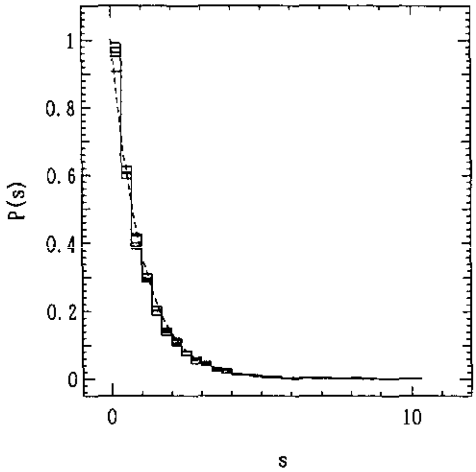


Figure 6. Nearest neighbour spacing distribution $P(s)$ for the length spectrum. The broken curve represents the Poisson distribution $P(s) = e^{-s}$.

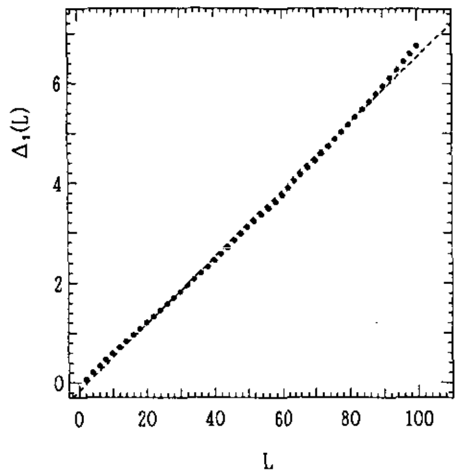


Figure 7. Spectral rigidity Δ_3 in the case of $\alpha_2 = \pi/2.4$.

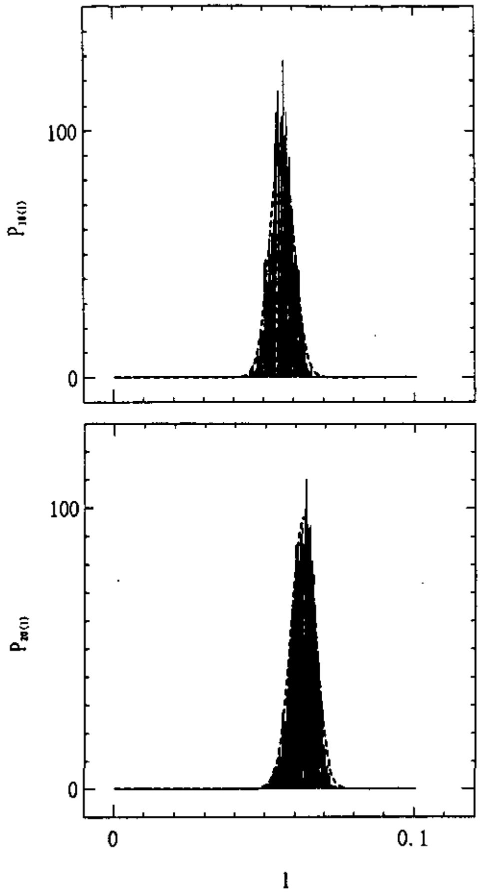
level statistics, the quadratic long range correlation, such as Δ_3- or Σ_2-statistics, is most familiar and it is easily accessible by numerical computations. To test the quadratic correlation of the length spectra, we here examined the spectral rigidity Δ_3 defined by

$$\Delta_3(L, x) \equiv \frac{1}{L} \min_{A,B} \int_{x-L/2}^{x+L/2} [n(\epsilon) - A\epsilon - B]^2 \, d\epsilon$$

where $n(\epsilon)$ is the unfolded cumulative density of states. If a sequence does not have any correlation, Δ_3 statistics should fit the straight line with the gradient 1/15. In the energy level sequence of classically integrable systems, it saturates in the long L regime. The reason for this saturation of the energy level sequence is that there exists a minimum length for periodic orbits [22]. On the other hand, as is actually shown in figure 7, numerical examinations demonstrate that the length spectrum of periodic orbits shows no sign of saturation, but it fits the straight line predicted for completely uncorrelated sequences well. For the cases with different system parameters, no noticeable difference is found. Therefore, we expect that there is no quadratic correlation between the length spectrum of hyperbolic billiard systems.

Another universality we expect is the distribution $P_n(l)$ of the length of the primitive periodic orbits with the reflection number n being fixed. Because nearest neighbour spacings of the length spectrum have no correlation and the length of l of a periodic orbit increases as the reflection times n increases, l can be expected to be randomly distributed around the mean length dependent on n. As shown in figure 8, for a sufficiently large number n, $P_n(l)$ can be fitted to the Gaussian distribution

$$P_n(l) = \frac{1}{\sqrt{2\pi n}\sigma} \exp\left(-\frac{(l - \bar{l}n)^2}{2\sigma^2 n}\right) \tag{5.1}$$



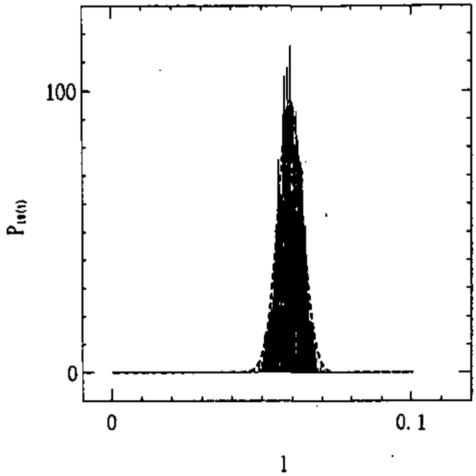


Figure 8. Length distributions for fixed reflection numbers in the case of $\alpha_2 = \pi/2.4$. The dotted curve denotes the Gaussian distributions.

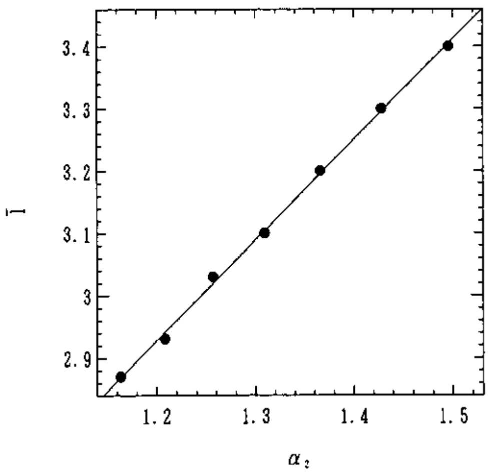


Figure 9. Scaled mean value of the length $\bar{l}(\alpha_2)$.

where values of $\bar{l}$ are given in figure 9. As α_2 decreases, the area of a billiard plane decreases, and thus the length of a periodic orbit corresponding to the same symbolic

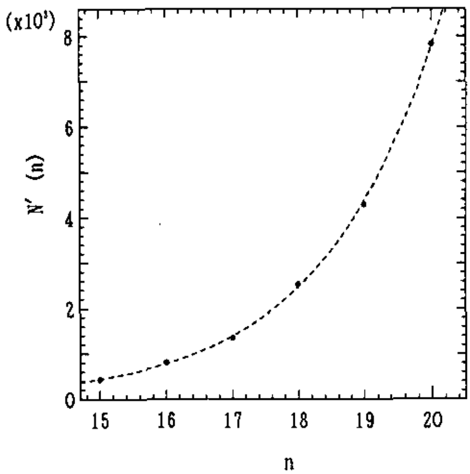


Figure 10. The number density $N'(n)$ of periodic orbits for reflection numbers in the case of $\alpha_2 = \pi/2.4$.

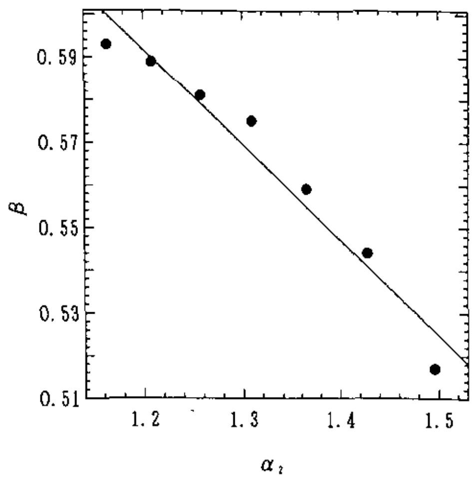


Figure 11. The relation between β and α_2.

sequence decreases. The cumulative density of the reflection times, $\sharp\{$ primitive periodic orbits γ with reflection times $n_\gamma \leqslant n\}$, is also very well approximated by constant $\times\ e^{\beta n}$ as shown in figure 10. The results of β for different values of parameter α_2 are shown in figure 11. The relation between α_2 and β is equivalent to that between α_2 and h. In the paper by Sieber and Steiner [9], the asymptotic form $e^{h'l}$ of $N(l)$ is derived from the asymptotic form $e^{\beta n}$ and Gaussian distribution of $P_n(l)$, using the stationary phase method, and the identity

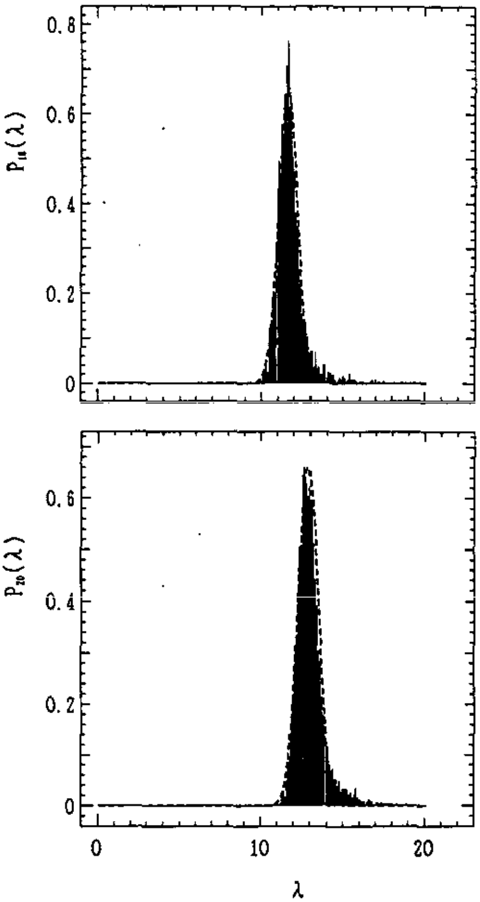
$$h' = \frac{1}{\sigma^2}\left(\bar{l} - \sqrt{\bar{l}^2 - 2\sigma^2\beta}\right)$$

is obtained. As shown in table 3, the difference between h' and h is less than 10%. Hence, these results are very consistent and confirm that there are no missing periodic orbits. In addition, these asymptotic forms are tested at several values of the system parameters. Therefore, they can be expected to be universal for hyperbolic billiards, at least dispersing ones.

Table 3. Comparison between topological entropy h and h'.

α_2	$\pi/2.1$	$\pi/2.2$	$\pi/2.3$	$\pi/2.4$	$\pi/2.5$	$\pi/2.6$	$\pi/2.7$
h	145.2	154.7	165.3	172.3	179.6	189.3	191.4
h'	153.9	167.5	177.3	190.9	197.2	205.9	210.5

The completely random nature of periodic orbits is revealed not only in their length but also their stability exponents. We examined the distribution $P'(\lambda)$ of the logarithms of the absolute values of the eigenvalues which describe the stability of the periodic orbits with the reflection times n being fixed. It is expected from the result for the distribution of the length l that the stability exponent is also uncorrelated.



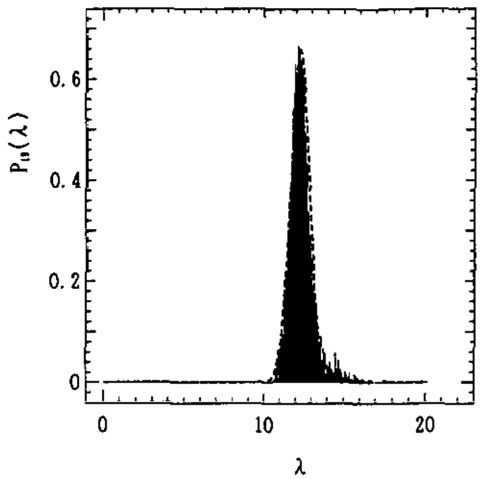


Figure 12. The stability exponent distributions for fixed reflection numbers in the case of $\alpha_2 = \pi/2.4$. The solid curve denotes the Gaussian distributions.

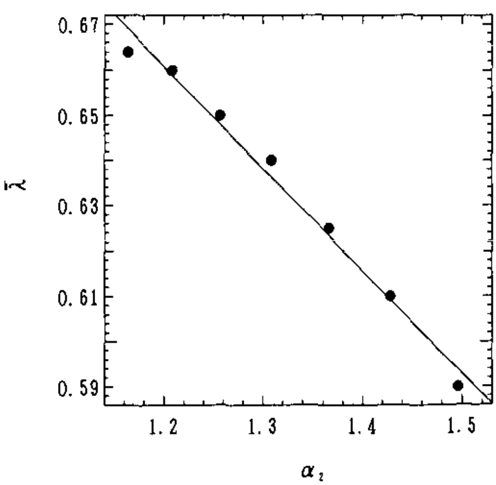


Figure 13. Scaled mean value of stability exponent $\bar{\lambda}(\alpha_2)$.

Figure 12 is the distribution of the stability exponent λ of the periodic orbits. As is clearly demonstrated in figure 12, the numerical histograms are also well fitted to the

Gaussian distribution

$$P_n'(\lambda) = \frac{1}{\sqrt{2\pi n}\sigma'} \exp\left(- \frac{(\lambda - \bar{\lambda} n)^2}{2\sigma'^2 n}\right) \tag{5.2}$$

where values of $\bar{\lambda}$ which corresponds to each α_2 are shown in figure 13. $\bar{\lambda}$ is the Lyapunov exponent of the system which measures the rate of divergence of neighbouring orbits or the chaoticity. Therefore, $\bar{\lambda}$ increases as α_2 decreases.

6. The semiclassical quantization

Full knowledge about the length spectrum of classical systems enables us to verify the validity of the Gutzwiller's semiclassical trace formula which relates the length spectrum to the energy spectrum of the corresponding quantum system [3]. In particular, knowing detailed information about each periodic orbit is necessary not only to study the convergency of the trace formula, but also to explain the universality of the short-range quantum level statistics through the trace formula. In the billiard version of Gutzwiller's formula, the density of states is represented only by the geometrical information of periodic orbits

$$\sum_n \delta(E - E_n) \approx \bar{d}(E) + \frac{1}{\pi\hbar} \mathrm{Re} \sum_\gamma \sum_{k=1}^{\infty} \frac{m l_\gamma}{p} \frac{1}{\sqrt{|2 - \mathrm{Tr} M_\gamma^k|}} \exp\left[\mathrm{i}k\left(\frac{p l_\gamma}{\hbar} - \frac{\nu_\gamma}{2}\pi\right)\right] \tag{6.1}$$

where $\bar{d}(E)$ denotes the mean level density, m represents the mass of a particle moving inside the billiard boundary and $p = \sqrt{2mE}$ is its momentum, γ denotes primitive periodic orbits and k repetition numbers. The phase ν_γ is the Maslov index which is twice the reflection times n_γ for Dirichlet boundary conditions. M_γ represents the linearized Poincaré map in the vicinity of γ, or the monodromy matrix which has eigenvalues $(-1)^{n_\gamma} \exp(\pm u_\gamma)$ $(u_\gamma > 0)$ in the case of dispersing billiards.

In the chaotic case, the right-hand side of equation (6.1) is not an absolutely convergent series since the proliferation rate of the number of periodic orbits overwhelms the decay rate of the amplitude factor [23]. The simplest method to make it converge is to introduce the convergence factor iϵ (ϵ is real positive) when the time-dependent propagator is transformed into the energy-dependent Green function in the derivation of Gutzwiller's formula. For the left-hand side of equation (6.1), this leads to the Lorenzian smoothing δ-function, and the damping factor is necessary for each term on the right-hand side. As a result, the trace formula has the form

$$\begin{aligned} &\sum_n \frac{1}{\pi} \frac{\epsilon}{(E - E_n)^2 + \epsilon^2} \\ &\qquad \approx \frac{1}{\pi\hbar} \mathrm{Re} \sum_\gamma \sum_{k=1}^{\infty} \frac{m l_\gamma}{p} \frac{1}{\sqrt{|2 - Tr M_\gamma^k|}} \exp\left[\mathrm{i}k\left(\frac{p l_\gamma}{\hbar} - \frac{\nu_\gamma}{2}\pi\right) - k\frac{\epsilon m}{\hbar p} l_\gamma\right]. \end{aligned} \tag{6.2}$$

Considering $N(l) = \text{constant} \times \exp(hl)$ and $u_\gamma = \bar{u}l$, the right-hand side series converges if the following condition is satisfied

$$\epsilon > \frac{\hbar p}{m}\left[h - \frac{\bar{u}}{2}\right] = \hbar\sqrt{\frac{2E}{m}}\left[h - \frac{\bar{u}}{2}\right]. \tag{6.3}$$

To see the validity of the semiclassical result, quantum energy levels are computed by solving Schrödinger's equation for Dirichlet boundary condition by means of the boundary element method. According to the inequality (6.3) and the preceding numerical results which are collected in figures 5, 9 and 13, if ϵ is chosen to be 10^5, the right-hand side of equation (6.2) in which the energy is less than 10^7 can converge. Results are shown in figure 14. It is safe to say that the semiclassical result, which includes the sum of several thousand primitive periodic orbits, is almost convergent as far as lower energy level are concerned. This is known by the fact that the shape of the curve is unchanged even if the number of periodic orbits is changed. For each parameter value, a remarkable agreement between the semiclassical and quantum density of states is observed for the lower 15 levels. In particular, it is surprising that a pair of levels with relatively small spacings can be gradually discriminated as the number of periodic orbits is increased.

7. Summary and discussion

Concluding this article, we summarize our results and pose unsolved problems in this area of study. We studied numerically periodic orbits of the dispersing billiards consisting of three circular arcs. A partial coding rule together with an efficient method of enumerating periodic orbits enables us to obtain a large number of unstable periodic orbits. The nearest neighbour spacing distribution of the length spectrum obeys Poisson law, and the distribution of length with a fixed bounce number shows Gaussian distribution. These results are consistent with those obtained by Sieber and Steiner [9]. Furthermore, we discovered that stability exponents with a fixed bounce number also shows Gaussian distribution. In addition, it is important that our results do not depend on the system parameters. These results strongly suggest that the length spectrum of hyperbolic billiard systems is highly uncorrelated. We conjecture that these properties of periodic orbits investigated in the present numerical analysis are universal at least for hyperbolic billiard systems.

On the other hand, the semiclassical density of states gives very good agreement with the exact quantum density of states even after an appropriate smoothing. Furthermore, as stated in section 6, the resulting semiclassical density of states is an almost convergent one. These indicate that a set of periodic orbits should have a certain kind of correlation or interrelation between them. However, at this stage, it is difficult to detect such a subtle correlation, although obtaining fine information about long periodic orbits is a necessary task in order to resum Gutzwiller's series. As is frequently emphasized, the most essential difficulty in Gutzwiller's formula lies in its convergency, which is a direct consequence of the exponential proliferation law. So far, two different approaches have been proposed to overcome it [18, 24]. The Riemann–Siegel look-alike formula does not rely on particular periodic orbits and is expected to work well in all systems. On the other hand, the curvature expansion method requires the symbolic organization of the periodic orbits, and works well

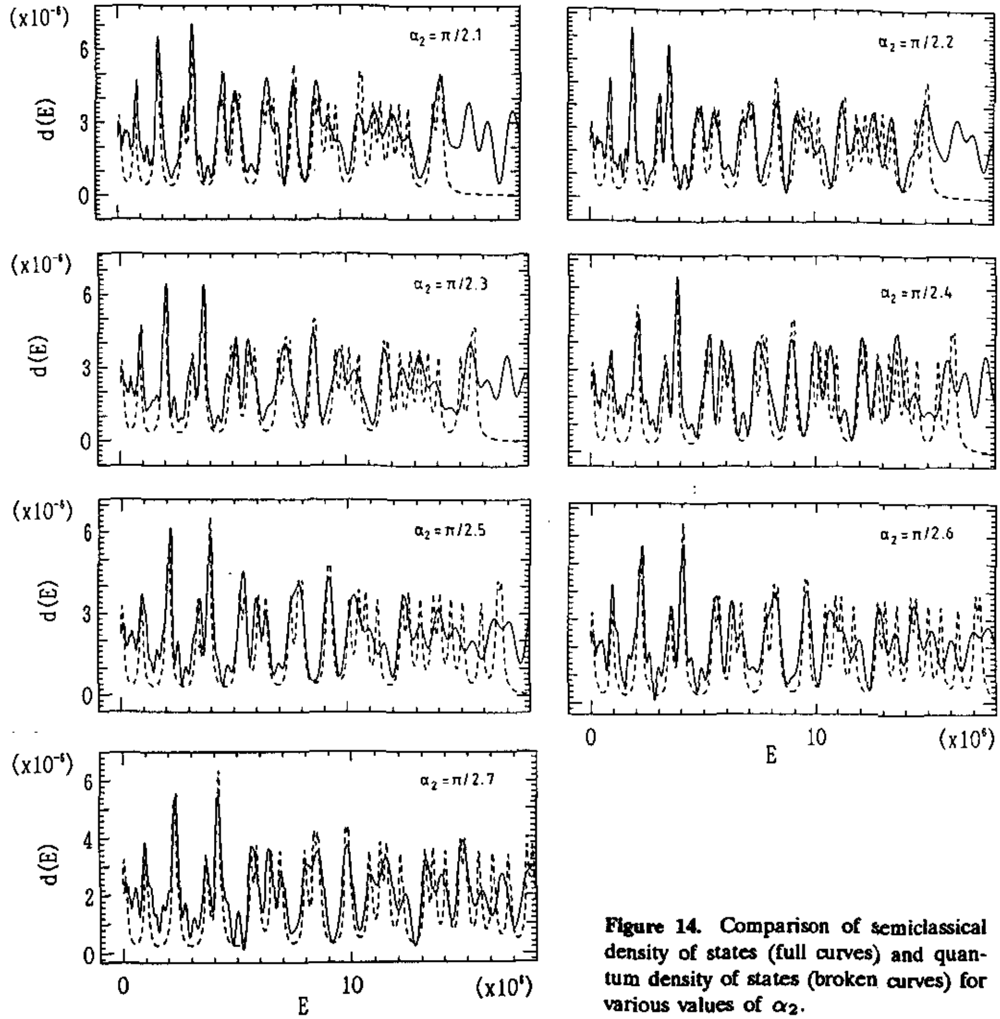


Figure 14. Comparison of semiclassical density of states (full curves) and quantum density of states (broken curves) for various values of α_2.

for systems having one-to-one correspondence between a sequence of codes and a periodic orbit [18]. The present dispersing billiard system is slightly complicated as compared with the three disk problem in the sense that any sequence of code does not necessarily have a corresponding periodic orbit. Hence, the present billiard possibly becomes a good candidate to check how far such a method works well even in the generic parameter regime, and where the complexity of partial coding breaks the convergency of resummed series. However, at present, this slight complication seems to bring us a great deal of difficulty in finding the correlation among periodic orbits. Every effort, for example, to investigate periodic orbits disappearing as a system parameter is gradually changed, has been unsuccessful. Moreover, all the periodic orbits seem to contribute equally to the semiclassical density of states. Attempts to exchange the order of the summation have not yielded different convergence rates. It does not appear that periodic orbits which survive under the change of the system parameter play any dominant roles in determining the gross features of the density of states. Nevertheless, there should be correlation between periodic orbits in order that Gutzwiller's formula can generate individual eigenstates. In particular, the presence of correlation is a necessary condition for the correct average density being given by the summation of very long orbits [25]. These unsolved questions must be dealt with in the future.

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References

[1] Arnold V I and Avez A 1968 *Ergodic Problems of Classical Mechanics* (New York: Benjamin)
[2] Berry M V 1987 *Proc. R. Soc.* A **413** 183
[3] Gutzwiller M C 1967 *J. Math. Phys.* **8** 1979; 1968 *J. Math. Phys.* **10** 1004; 1970 *J. Math. Phys.* **11** 1791; 1971 *J. Math. Phys.* **12** 343
[4] Aurich R, Bogomolony E B and Steiner F 1991 *Physica* **48D** 91
[5] O'Connor P W and Tomsovic S 1991 *Ann. Phys., NY* **207** 218
[6] Keating J 1991 *Nonlinearity* **4** 277, 309
[7] Ozorio de Almeida A M and Saraceno M 1991 *Ann. Phys., NY* **210** 1
[8] Wintgen D 1988 *Phys. Rev. Lett.* **61** 1803
[9] Sieber M and Steiner F 1990 *Physica* **44D** 248; 1990 *Phys. Lett.* **144A** 415, 159
[10] Bunimovich L A 1989 *Dynamical Systems II* ed Ya G Sinai (Berlin: Springer) p 151
[11] Machta J 1983 *J. Stat. Phys.* **32** 555
[12] Sinai Ya G 1976 *An Introduction to Ergodic Theory* (Moscow: Erivan(Lecture 10))
[13] Richens P J and Berry M V 1981 *Physica* **2D** 495
[14] Sinai Ya G 1970 *Russ. Math. Surv.* **25** 137
[15] Gallavotti G and Ornstein G 1974 *Commun. Math. Phys.* **38** 83
[16] Bunimovich L A, Sinai Ya G and Chernov N I 1980 *Commun. Math. Phys.* **73** 247; 1986 *Commun. Math. Phys.* **107** 357
[17] Bunimovich L A, Sinai Ya G and Chernov N I 1990 *Russ. Math. Surv.* **45** 105
[18] Cvitanovic P and Eckhardt B 1989 *Phys. Rev. Lett.* **63** 823
[19] Motita T 1991 *Trans. Am. Math. Soc. 1991* **325** 819
[20] Gaspard P and Rice S A 1989 *J. Chem. Phys.* **90** 2225, 2242, 2255
[21] Chen Q, Meiss J D and Percival I C 1987 *Physica* **29D** 143
[22] Berry M V 1985 *Proc. R. Soc.* **400** 229
[23] Eckhardt B and Aurell E 1989 *Europhys. Lett.* **9** 509
[24] Berry M V and Keating J 1990 *J. Phys. A: Math. Gen.* **23** 4839
Keating J 1992 *Proc. R. Soc.* A **436** 99
[25] Keating J 1991 *Quantum Chaos* ed H A Cerderia *et al* (Singapore: World Scientific)